

BOUNDED STATES OF ONE DIMENSIONAL SCHRODINGER SYSTEMS

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ABSTRACT. In this paper, we study the existence problem of bound states of one dimensional Schrodinger system via the blow-up method.

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1. INTRODUCTION

The coupled nonlinear Schrodinger (NLS) equations arise naturally in nonlinear optics and they are also considered as the model of Bose-Einstein condensate. As a interesting physical model, the planar stationary light beams propagating in the z -direction in a nonlinear medium are described by an NLS equation of the form

$$(1) \quad iE_z + E_{xx} + k|E|^2E = 0$$

where i denotes the imaginary unit, $E = E(x, z)$ denotes the complex envelope of an electric field and E_{xx} is its second partial derivative, and k is the coupling constant, which is assumed to be positive and corresponds physically self-focusing of the medium. If we are looking for the bounded states (or standing waves), namely the special solutions to the system above in the form

$$E_j(z, x) = e^{i\lambda z} u_j(x), \quad \lambda > 0,$$

we are led to solve the elliptic system of the form

$$-u_{xx} + \lambda u = k|u|^2u, \quad x \in \mathbf{R}$$

where $u = (u^1, u^2)$ is the unknown vector valued function on the real line \mathbf{R} . The cubic nonlinear term on the right hand side makes the system possible to be solved explicitly. When the physical situation is more complicated, the coupling constant k can be changed into a positive function $Q(t)$ and we are led to consider the elliptic system

$$-u_{xx} + \lambda u = Q(t)|u|^2u, \quad x \in \mathbf{R}.$$

Let $Q(t) = Q(|t|) \geq 0$ be a bounded even function on \mathbf{R} . Let $a > 0$ be a positive constant. Let $p > 1$. Assume that $Q(0) > 0$. We shall consider the non-trivial non-negative solutions to the following elliptic system

$$(2) \quad -u_{tt} + au = Q(t)|u|^{p-1}u, \quad x \in \mathbf{R}.$$

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The study of the problem above is non-trivial in the sense that there is no Sobolev imbedding theorem on the whole real line. This fact is overlooked by some previous authors.

There are some results concerning with the special case when $p = 3$ by the perturbation method, see [1]. The higher dimensional cases have studied by Lin-Wei [4], Ma-Zhao [6] and others [2], [5],[3], [7].

We set up the following result.

Theorem 1. *There is at least one non-negative non-trivial bounded solution u to (2).*

The proof of this result is by the blow-up method. Roughly speaking, it is a proof of arguing by contradiction, which is given in next section. A general result can be proved similarly and is stated in the last section.

2. THE PROOF OF THEOREM 1

For any $R > 0$, we denote $I_R = [-R, R]$. Define on $H_R = H_0^1(I_R)$, the functional

$$E_R(u) = \int_{I_R} \frac{1}{2}(|u'|^2 + au^2) - \frac{1}{p+1} \int_{I_R} Q(t)|u|^{p+1}.$$

Its Euler-Lagrange equation is

$$-u'' + au = Q(t)|u|^{p-1}u, \text{ in } I_R$$

with $u = 0$ at $t = R$ and $-R$.

Introduce the Nehari manifold

$$\mathbf{N} = \{u \in H_R - \{0\}; \int_{I_R} (|u'|^2 + au^2) = \int_{I_R} Q(t)|u|^{p+1}\}.$$

Restricted to \mathbf{N} the functional E_R can be written as

$$E_R(u) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{I_R} (|u'|^2 + au^2).$$

Then by the direct method we know that there is a minimizer $u_R \in \mathbf{N}$ to the minimization problem

$$d_R := \int_{\mathbf{N}} E_R(u) > 0.$$

Furthermore, $u_R > 0$ in the sense that each component is positive. In fact using the Sobolev inequality we know that

$$\int_{I_R} Q(t)|u|^{p+1} \leq C \left[\int_{I_R} (|u'|^2 + au^2) \right]^{(p+1)/2}.$$

For $u \in \mathbf{N}$, we have

$$\int_{I_R} (|u'|^2 + au^2) = \int_{I_R} Q(t)|u|^{p+1}.$$

Then we have

$$\int_{I_R} (|u'|^2 + au^2) \leq C \left[\int_{I_R} (|u'|^2 + au^2) \right]^{(p+1)/2}.$$

This implies that

$$\int_{I_R} (|u'|^2 + au^2) \geq C(R) > 0$$

for some uniform constant $C(R) > 0$. Hence $d_R \geq C(R) > 0$. We may assume that each component is non-trivial, otherwise, we just delete the trivial functions.

Using the symmetry property we know that each component of u_R has its maximum point at origin. For otherwise the component will be trivial and then it is zero.

Without loss of generality we assume that the maximum component of u_R is the first one in the sense that

$$u_R^1(0) = \max_{I_R} u_R^1(t) = M_R \geq u_R^j(0) > 0.$$

Let $R = R_j \rightarrow \infty$. Let $M_j = M_{R_j}$ and $u_j = u_{R_j}$. We *claim* that M_j is uniformly bounded and then we have proved Theorem 1 by taking the limit. Assume that $M_j \rightarrow \infty$, which will be showed that it is impossible. We define the blow-up sequence

$$v_j(t) = M_j^{-1} u_j(M_j^\beta t)$$

for $\beta > 0$ to be determined later. Let

$$Q_j(t) = Q(M_j^\beta t).$$

Note that

$$v_j''(t) = M_j^{2\beta-1} u_j''$$

and

$$|v_j|^{p-1} v_j = M_j^{-p} |u_j|^{p-1} u_j.$$

We let $\beta = \frac{1-p}{2} < 0$. Since v_j is uniformly bounded, we may use the elliptic regularity theory to conclude that (v_j) has a convergent subsequence (v_{j_k}) in C_{loc}^3 , which is still denoted by (v_j) and its C^2 limit \bar{v} such that

$$(3) \quad -\bar{v}'' = Q(0)\bar{v}^p, \text{ in } (-\infty, \infty)$$

with $\bar{v}^1(0) = 1$.

We shall prove that (3) has no nontrivial positive solution. Let $f = \bar{v}^1$. by maximum principle we know that $f > 0$ in $(-\infty, \infty)$. *Claim* that $f' > 0$. If not, then for some t_0 , we have

$$f'(t_0) \leq 0.$$

Then we have for $t > t_0$,

$$f'(t) = f'(t_0) + f''(\bar{t})(t - t_0) < 0.$$

Here we have used $f''(\bar{t}) = Q(0)|\bar{v}|^{p-1} f < 0$.

For any $t > t_1 > t_0$, we have

$$f'(t) < f'(t_1) < 0.$$

Hence we have

$$f(t) = f(t_1) + \int_{t_1}^t f'(s) ds \leq f(t_1) + f'(t_1)(t - t_1) \rightarrow -\infty.$$

as $t \rightarrow \infty$, a contradiction.

We remark that the same argument works for other component. Then we have $|\bar{v}|^{p-1}f(t) \geq |\bar{v}|^{p-1}f(s)$ for $t > s$.

Then,

$$-f''(t) = Q(0)|\bar{v}|^{p-1}f(t) \geq Q(0)|\bar{v}|^{p-1}f(s) > 0.$$

We then conclude that

$$f'(s) > f'(s) - f'(t) = Q(0) \int_s^t |\bar{v}|^{p-1}f(\tau)d\tau \geq Q(0)|\bar{v}|^{p-1}f(s)(t-s) \rightarrow \infty$$

as $t \rightarrow \infty$, which is impossible.

3. REMARKS ABOUT RELATED SYSTEMS

One may also look for bounded states of the form

$$E_j(z, x) = e^{i\lambda_j z} u_j(x), \quad \lambda_j > 0,$$

to the system (1) and the reduced system is of the form

$$(4) \quad -u_{xx}^j + \lambda_j u^j = Q(t)|u|^{p-1}u^j, \quad x \in \mathbf{R}.$$

Here $p > 1$ and $\lambda_j > 0$, $j = 1, \dots, N$.

Then using similar method to theorem 1, we may use the functional

$$E_R(u) = \int_{I_R} \frac{1}{2}(|u'|^2 + \sum_j \lambda_j (u^j)^2) - \frac{1}{p+1} \int_{I_R} Q(t)|u|^{p+1}$$

to obtain the following result.

Theorem 2. Assume that $Q(t) = Q(|t|) \geq 0$ is a bounded function in \mathbf{R} with $Q(0) > 0$ and $\lambda_j > 0$ for all $j = 1, \dots, N$. There is at least one non-negative non-trivial bounded solution $u = (u^1, \dots, u^N)$ to the system (4).

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